SWAIM, TERRYL SILER. On Hereditary Torsion Theories. (1973)
Directed by: Dr. Robert L. Bernhardt. Pp. 27.

This thesis deals with S. E. Dickson's concept of a hereditary torsion theory for the category $_R M$ of left R-modules over an arbitrary ring R. It is proved that a class of left R-modules is a hereditary torsion class if and only if there is an injective module which generates it. A method is given for generating a torsion-torsionfree class from a projective module. Further, it is proved that if R is a semi-perfect ring, then under certain conditions a torsion-torsionfree class is generated by a projective module. A characterization of a hereditary torsion class in terms of a module X uniquely determined by the elements of the torsion filter is given. It is proved that there is a one-to-one correspondence between the two-sided, idempotent ideals of R and the torsion-torsionfree classes for $_R M$. The thesis concludes with the definition of a centrally splitting torsion theory, and with the proof that there is a one-to-one correspondence between the central idempotents of R and the centrally splitting torsion theories for $_R M$.

ON HEREDITARY TORSION THEORIES

by

Terryl Siler Swaim

A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
May, 1973

Approved by

Robert L. Bernhardt
Thesis Adviser

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Adviser Robert L. Bernhardt

Oral Examination Committee Members William P. Love

E.E. Posey

Karl Ray Gentry

April 4, 1973
Date of Examination

ACKNOWLEDGMENT

The author wishes to sincerely thank Dr. Robert L. Bernhardt for his assistance and guidance in the preparation of this thesis, and especially for his patience and encouragement when these were needed so much.

TABLE OF CONTENTS

INTRODUCTION

The concepts of torsion and torsionfree originated in abelian group theory. In [3] Dickson defines a torsion theory for an arbitrary abelian category. Since the category ${}_R\mathcal{M}$ of left R-modules over an arbitrary ring R is an abelian category, Dickson's concept of a torsion theory can be specified for this category. This thesis will deal with the concepts of torsion and torsionfree as they are thus specified in ${}_R\mathcal{M}$.

In Chapter I, a foundation is formed for the study of torsion theories for ${}_R\mathcal{M}$. The definition of a torsion theory for ${}_R\mathcal{M}$ is given, and several theorems which give characterizations of torsion theories are stated. A torsion filter is defined for an arbitrary ring R, and a one-to-one correspondence between torsion filters for R and hereditary torsion theories for ${}_R\mathcal{M}$ is given. A torsion-torsionfree (TTF) theory is defined, and some important properties of the associated torsion filter are stated.

Chapter II deals with the generation of torsion theories. It is shown that a class $\mathcal{T}$ of left R-modules is a hereditary torsion class if and only if there is some injective module Y such that $\mathcal{T} = \{M \in {}_R\mathcal{M} | \mathrm{Hom}(M,Y) = 0\}$. If P is a projective module, it is proved that $\mathcal{T} = \{M \in {}_R\mathcal{M} | \mathrm{Hom}(P,M) = 0\}$ is a TTF class. Two forms of the converse of this theorem are given when R is stipulated to be a semi-perfect ring: one form where the TTF class is stable, and one form where the torsionfree class associated with the TTF class is closed under homomorphic images.

A concluding theorem gives a characterization of a hereditary torsion theory in terms of a module X uniquely determined by the elements of the torsion filter $F(T)$. For the special case of a TTF theory, the module X can be greatly simplified.

In Chapter III, TTF classes are studied in greater depth. A useful characterization of a TTF theory is given: It is proved that if I is an idempotent, two-sided ideal of R, then $T = \{M \in {}_R M \mid IM = 0\}$ is a TTF class and I is the unique smallest element of $F(T)$. Using this result, a one-to-one correspondence between idempotent, two-sided ideals of R and TTF classes for ${}_R M$ is demonstrated. A centrally splitting torsion theory is defined in terms of a theorem listing equivalent properties of a special type of TTF class. It is further shown that for any centrally splitting torsion theory (T, F), there is a central idempotent e of R such that $T = \{M \in {}_R M \mid (1-e)M = M\}$ and $F = \{N \in {}_R M \mid eN = N\}$. Finally, it is proved that there exists a one-to-one correspondence between the centrally splitting torsion theories for ${}_R M$ and the central idempotents of R.

Throughout this thesis, the term "ring" will mean an associative ring with unity 1, and such a ring will be denoted by R. All modules will be unital left R-modules, and the notation ${}_R M$ will be used to denote the category of all unital left R-modules. The symbol $\mathrm{Hom}_R(M,N)$ will denote the set of all R-homomorphisms from a module ${}_R M$ into a module ${}_R N$. Specific mention of the ring R will be omitted, unless needed to prevent confusion. Thus M will be used instead of ${}_R M$, Hom(M,N) instead of $\mathrm{Hom}_R(M,N)$, and so forth.

If the module N is a left R-submodule of the module M, this will be denoted by $N \leq M$. Since every submodule I of ${}_R R$ is also a left ideal of

R, $I \leq R$ means I is a left ideal of R. If $N \leq M$ and $x \in M$, then $(N:x) = \{r \in R \mid rx \in N\}$. It can easily be shown that $(N:x)$ is a left ideal of R.

Let $f \in \mathrm{Hom}(M,N)$. Then $\mathrm{Im}\, f = \{f(m) \mid m \in M\}$, and $\mathrm{Ker}\, f = \{m \in M \mid f(m) = 0\}$. It can be shown that $\mathrm{Im}\, f \leq N$ and $\mathrm{Ker}\, f \leq M$.

A sequence $\ldots \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} M_3 \xrightarrow{f_4} \ldots$ of modules and module homomorphisms is said to be <u>exact</u> provided $\mathrm{Ker}\, f_i = \mathrm{Im}\, f_{i-1}$ for every i. Thus the sequence $0 \to M \xrightarrow{f} N$ is exact provided f is one-to-one, and the sequence $M \xrightarrow{f} N \to 0$ is exact provided f is onto.

A module P is said to be <u>projective</u> provided if $M \xrightarrow{f} N \to 0$ is an exact sequence of modules, and if $g \in \mathrm{Hom}(P,N)$, then there exists $h \in \mathrm{Hom}(P,M)$ such that $f \circ h = g$. A module Q is said to be <u>injective</u> provided if $0 \to M \xrightarrow{f} N$ is an exact sequence of modules, and if $g \in \mathrm{Hom}(M,Q)$, then there exists $h \in \mathrm{Hom}(N,Q)$ such that $h \circ f = g$.

If $N \leq M$ we say that N is <u>essential</u> in M, denoted $N \underline{\Delta} M$, provided for all $0 \neq L \leq M$, $N \cap L \neq 0$. If $N \underline{\Delta} M$ we say that M is an <u>essential extension</u> of N. For every module M there exists a smallest injective module that is an essential extension of M, called the <u>injective envelope</u> of M and denoted by $E_R(M)$, or simply $E(M)$.

Throughout this thesis we will refer to several "closure" properties of a class of modules. Let $\mathcal{A}$ be a class of modules. Then

(a) $\mathcal{A}$ is <u>closed under homomorphic images</u> if $M \in \mathcal{A}$ and $M \to N \to 0$ exact implies $N \in \mathcal{A}$;

(b) $\mathcal{A}$ is <u>closed under submodules</u> if $M \in \mathcal{A}$ and $0 \to N \to M$ exact implies $N \in \mathcal{A}$;

(c) $\mathcal{A}$ is <u>closed under extensions</u> if $M,N \in \mathcal{A}$ and $0 \to M \to B \to N \to 0$ exact implies $B \in \mathcal{A}$;

(d) A is <u>closed under direct sums</u> (<u>direct products</u>) if $A_i \in A$ for each $i \in I$ implies $\oplus\Sigma_I A_i \in A$ ($\Pi_I A_i \in A$);

(e) A is <u>closed under essential extensions</u> if $M \in A$ and $M \trianglelefteq N$ implies $N \in A$;

(f) A is <u>closed under injective envelopes</u> if $M \in A$ implies $E(M) \in A$.

The conclusion of a proof will be indicated by $\square$.

CHAPTER I

PRELIMINARIES

The following definitions and theorems form a foundation for further investigation into torsion theories. All of the following, through Theorem 1.6, is taken from the paper by Dickson [3]. The reader will find all of these results in this chapter developed in the (unpublished) master's thesis of Bennett [1]. We will only state, without proof, the theorems in this chapter, since they are not the subject of investigation in this thesis.

1.1 DEFINITION. A torsion theory for ${}_R\mathcal{M}$ is a pair $(\mathcal{T},\mathcal{F})$ of classes of left R-modules which satisfy the following properties:

(a) $\mathcal{T} \cap \mathcal{F} = 0$;

(b) $\mathcal{T}$ is closed under homomorphic images;

(c) $\mathcal{F}$ is closed under submodules;

(d) For each $M \in {}_R\mathcal{M}$ there exists a submodule M_t of M such that $M_t \in \mathcal{T}$ and $M/M_t \in \mathcal{F}$.

If $(\mathcal{T},\mathcal{F})$ is a torsion theory for ${}_R\mathcal{M}$, $\mathcal{T}$ is called a torsion class and $\mathcal{F}$ is called a torsionfree class. If $\mathcal{T}$ is also closed under submodules, it is called a hereditary torsion class.

It can be easily verified that if $T \in \mathcal{T}$ and $M \in {}_R\mathcal{M}$ such that $M \cong T$, then $M \in \mathcal{T}$, for $T \to M \to 0$ is an exact sequence and $\mathcal{T}$ is closed under homomorphic images. Similarly, if $F \in \mathcal{F}$ and $N \in {}_R\mathcal{M}$ such that $N \cong F$, then $N \in \mathcal{F}$.

<u>1.2 THEOREM</u>. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. Then $\mathcal{T}$ and $\mathcal{F}$ uniquely determine each other. Specifically, $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}\}$ and $\mathcal{F} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,M) = 0$ for all $T \in \mathcal{T}\}$.

<u>1.3 THEOREM</u>. A class $\mathcal{T}$ of left R-modules is a torsion class if and only if $\mathcal{T}$ is closed under (a) homomorphic images, (b) extensions, and (c) arbitrary direct sums.

<u>1.4 THEOREM</u>. A class $\mathcal{F}$ of left R-modules is a torsionfree class if and only if $\mathcal{F}$ is closed under (a) submodules, (b) extensions, and (c) arbitrary direct products.

<u>1.5 COROLLARY</u>. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. Then for each $M \in {}_R\mathcal{M}$, $M_t = \Sigma\{T \le M \mid T \in \mathcal{T}\}$ and $M_t = \cap\{K \le M \mid M/K \in \mathcal{F}\}$.

It is clear from the above characterization of M_t that M_t is the unique largest submodule of M contained in $\mathcal{T}$. That is, if $N \le M$ such that $N \in \mathcal{T}$, then $N \subseteq M_t$.

<u>1.6 THEOREM</u>. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. Then $\mathcal{T}$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.

<u>1.7 DEFINITION</u>. A set $\mathcal{B}$ of left ideals of R is called a <u>torsion filter</u> for R provided $\mathcal{B} \ne \phi$ and

(a) If $I \in \mathcal{B}$ and $I \le I' \le R$, then $I' \in \mathcal{B}$;

(b) If $I, I' \in \mathcal{B}$, then $I \cap I' \in \mathcal{B}$;

(c) If $I \in \mathcal{B}$, then $(I:a) \in \mathcal{B}$ for all $a \in R$;

(d) If $I \le R$ and if there exists $I' \in \mathcal{B}$ such that $(I:a) \in \mathcal{B}$ for all $a \in I'$, then $I \in \mathcal{B}$.

One can verify that (d) implies the following: if $I, I' \in \mathcal{B}$, then $I \cdot I' \in \mathcal{B}$. For $I \subseteq \{r \in R \mid ra \in I \cdot I'\} = (I \cdot I':a)$ for all $a \in I'$, so $(I \cdot I':a) \in \mathcal{B}$ for all $a \in I'$ and by (d) $I \cdot I' \in \mathcal{B}$.

1.8 THEOREM. There is a one-to-one correspondence between hereditary torsion classes and torsion filters for R. Specifically, if $\mathcal{T}$ is a hereditary torsion class, then the corresponding torsion filter, denoted $F(\mathcal{T})$, is $\{I \leq R \mid R/I \in \mathcal{T}\}$, and if $\mathcal{B}$ is a torsion filter for R, then the corresponding hereditary torsion class is $\{M \in {}_R\mathcal{M} \mid (0:x) \in \mathcal{B}$ for all $x \in M\} = \{M \in {}_R\mathcal{M} \mid$ for all $x \in M$, $Ix = 0$ for some $I \in \mathcal{B}\}$. [4]

The following results on torsion-torsionfree classes are taken from the paper by Jans [5].

1.9 DEFINITION. A class $\mathcal{T}$ of left R-modules is a torsion-torsionfree class, or a TTF class, provided $\mathcal{T}$ is closed under homomorphic images, submodules, extensions, arbitrary direct sums, and arbitrary direct products. If $\mathcal{T}$ is a TTF class, then there exist a torsionfree class $\mathcal{F}$ and a torsion class $\mathcal{C}$ such that $(\mathcal{T},\mathcal{F})$ and $(\mathcal{C},\mathcal{T})$ are torsion theories for ${}_R\mathcal{M}$. When $\mathcal{T}$ is a TTF class, then $(\mathcal{C},\mathcal{T},\mathcal{F})$ is called a TTF theory for ${}_R\mathcal{M}$.

Since each TTF class $\mathcal{T}$ is a hereditary torsion class, there exists a torsion filter $F(\mathcal{T})$ corresponding to $\mathcal{T}$. The next theorem gives an important property of this filter.

1.10 THEOREM. $\mathcal{T}$ is a TTF class if and only if $F(\mathcal{T})$ has a unique smallest element.

One can show that given a TTF theory $(\mathcal{C},\mathcal{T},\mathcal{F})$, the unique smallest element of $F(\mathcal{T})$ is R_c, the $\mathcal{C}$-torsion submodule of R, and that both R_c and R_t are two-sided ideals of R. Furthermore, R_c is idempotent, by which we mean that $R_c{}^2 = R_c$.

CHAPTER 2

GENERATING TORSION THEORIES

The following theorem shows that every hereditary torsion class can be generated by some injective module, and gives a method for generating a hereditary torsion class when given an injective module.

2.1 THEOREM. Let $\mathcal{T}$ be a class of left R-modules. Then $\mathcal{T}$ is a hereditary torsion class if and only is there exists an injective module Y such that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,Y) = 0\}$.

Proof: $(\rightarrow)$ Assume $\mathcal{T}$ is a hereditary torsion class. Then there exists a torsionfree class $\mathcal{F}$ such that $(\mathcal{T},\mathcal{F})$ is a torsion theory for ${}_R\mathcal{M}$. Let $\mathcal{L}$ be a set of cyclic modules in $\mathcal{F}$ such that every cyclic module in $\mathcal{F}$ is isomorphic to one and only one element of $\mathcal{L}$. Let $A = \Pi\{L \mid L \in \mathcal{L}\}$, and let $Y = E(A)$. Since $\mathcal{F}$ is closed under arbitrary direct products, $A \in \mathcal{F}$. Since $\mathcal{T}$ is hereditary, $\mathcal{F}$ is closed under injective envelopes, so $Y \in \mathcal{F}$. Let $\mathcal{T}' = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,Y) = 0\}$. Clearly $\mathcal{T} \subseteq \mathcal{T}'$, since $\mathrm{Hom}(T,Y) = 0$ for all $T \in \mathcal{T}$. Let $M \in \mathcal{T}'$, $F \in \mathcal{F}$, and $f \in \mathrm{Hom}(M,F)$. Let $x \in f(M) \subseteq F$. Then $Rx \leq F$, so $Rx \in \mathcal{F}$. Since Rx is cyclic, there exists $L \in \mathcal{L}$ such that $Rx \simeq L$, so there exists an isomorphism $g \in \mathrm{Hom}(L,Rx)$. Let $\theta_L \in \mathrm{Hom}(L,A)$ be the natural injection, and let $i \in \mathrm{Hom}(Rx,F)$ and $i_A \in \mathrm{Hom}(A,Y)$ be the inclusion mappings. Then we have the following diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r, dashed] & L \arrow[r, dashed, "g"] \arrow[d, dashed, "\theta_L"'] & Rx \arrow[r, dashed, "i"] & F \\ & A \arrow[d, dashed, "i_A"'] & & \\ & Y & & \end{tikzcd}$$

Since Y is injective, there exists h ε Hom(F,Y) such that this diagram commutes. Since x ε f(M), there exists m ε M such that f(m) = x. Now hf ε Hom(M,Y) = 0, so hf = 0. Then h(x) = hf(m) = 0, so 0 = hig(L) = $i_A\theta_L$(L). Since i_A and θ_L are one-to-one, this implies L = 0. Then Rx = 0, so x = 0. Thus f(M) = 0, so f = 0, and Hom (M,F) = 0. Therefore M ε T, $T' \subseteq T$, and $T = \{M \varepsilon\ _RM \mid \text{Hom}(M,Y) = 0\}$.

($\leftarrow$) Let Y be an injective module and $T = \{M \varepsilon\ _RM \mid \text{Hom}(M,Y) = 0\}$. To show that T is a hereditary torsion class we must show that T is closed under (a) homomorphic images, (b) submodules, (c) extensions, and (d) arbitrary direct sums.

(a) Let T ε T and let $T \xrightarrow{f} M \rightarrow 0$ be exact. Let g ε Hom(M,Y). then gf ε Hom(T,Y) = 0, so gf = 0. Let m ε M. Then there exists t ε T such that f(t) = m, so g(m) = gf(t) = 0. Thus g = 0, Hom(M,Y) = 0, and M ε T. Thus T is closed under homomorphic images.

(b) Let T ε T and let $0 \rightarrow M \xrightarrow{f} T$ be exact. Let g ε Hom(M,Y). Since Y is injective there exists h ε Hom(T,Y) such that the diagram

$$\begin{tikzcd} 0 \arrow[r, dashed] & M \arrow[r, dashed, "f"] \arrow[d, dashed, "g"'] & T \arrow[dl, dashed, "h"] \\ & Y & \end{tikzcd}$$

commutes. Let m ε M. Now Hom(T,Y) = 0, so h = 0, and g(m) = hf(m) = h(f(m)) = 0. Then g = 0, Hom(M,Y) = 0, and M ε T. Thus T is closed under submodules.

(c) Let $0 \to T_1 \to B \to T_2 \to 0$ be exact with T_1, $T_2 \in \mathcal{T}$. It can be shown that $0 \to \mathrm{Hom}(T_2,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(T_1,Y)$ is exact under suitable functions. Since $\mathrm{Hom}(T_2,Y)$ and $\mathrm{Hom}(T_1,Y)$ are both zero, we have that $\mathrm{Hom}(B,Y) = 0$. Thus $B \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions.

(d) Let $\{T_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. It can be shown that $\mathrm{Hom}(\oplus\Sigma_I T_i,Y) \simeq \Pi_I \mathrm{Hom}(T_i,Y) = 0$, so $\mathrm{Hom}(\oplus\Sigma_I T_i,Y) = 0$ and $\oplus\Sigma_I T_i \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under arbitrary direct sums.

Therefore, $\mathcal{T}$ is a hereditary torsion class. □

The next theorem shows that, given a projective module, one can generate a TTF class.

<u>2.2 THEOREM</u>. Let P be a projective module. Then $\mathcal{T} = \{M \in {}_R M \mid \mathrm{Hom}(P,M) = 0\}$ is a TTF class.

<u>Proof</u>: To show that $\mathcal{T}$ is a TTF class we must show that $\mathcal{T}$ is closed under (a) homomorphic images, (b) submodules, (c) extensions, (d) arbitrary direct products, and (e) arbitrary direct sums.

(a) Let $T \in \mathcal{T}$ and let $T \xrightarrow{f} M \to 0$ be exact. Let $g \in \mathrm{Hom}(P,M)$. Since P is projective, there exists $h \in \mathrm{Hom}(P,T)$ such that the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, dashed, "g"] & \\ T \arrow[r, dashed, "f"] & M \arrow[r, dashed] & 0 \end{tikzcd}$$

commutes. Since $\mathrm{Hom}(P,T) = 0$, then $h = 0$. Let $p \in P$. Then $g(p) = fh(p) = f(h(p)) = f(0) = 0$, so $g = 0$, $\mathrm{Hom}(P,M) = 0$ and $M \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under homomorphic images.

(b) Let $T \in \mathcal{T}$ and let $0 \to M \xrightarrow{f} T$ be exact. Let $g \in \mathrm{Hom}(P,M)$. Then $fg \in \mathrm{Hom}(P,T) = 0$, so $fg = 0$. Let $p \in M$. Then $0 = fg(p) = f(g(p))$, and since f is one-to-one, $g(p) = 0$. Thus $g = 0$, $\mathrm{Hom}(P,M) = 0$ and $M \in \mathcal{T}$. Therefore $\mathcal{T}$ is closed under submodules.

(c) Let $0 \to T_1 \to B \to T_2 \to 0$ be exact with $T_1, T_2 \in \mathcal{T}$. It can be shown that $\mathrm{Hom}(P,T_1) \to \mathrm{Hom}(P,B) \to \mathrm{Hom}(P,T_2) \to 0$ is exact under suitable functions. Since $\mathrm{Hom}(P,T_1)$ and $\mathrm{Hom}(P,T_2)$ are both zero, we have that $\mathrm{Hom}(P,B) = 0$. Then $B \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions.

(d) Let $\{T_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. It can be shown that $\mathrm{Hom}(P,\Pi_I T_i) \simeq \Pi_I \mathrm{Hom}(P,T_i) \simeq 0$, so $\mathrm{Hom}(P,\Pi_I T_i) = 0$ and $\Pi_I T_i \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under arbitrary direct products.

(e) Let $\{T_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. Since $\oplus\Sigma_I T_i$ is a submodule of $\Pi_I T_i \in \mathcal{T}$, then $\oplus\Sigma_I T_i \in \mathcal{T}$ by (b). Thus $\mathcal{T}$ is closed under arbitrary direct sums. $\square$

We are interested in the converse to Theorem 2.2; that is, given a TTF class $\mathcal{T}$, when can we find a projective module P such that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$? In order to obtain results of this nature, it is necessary to restrict both the type of ring and the type of TTF class under consideration. We shall seek to do this with a minimum of technical definitions.

2.3 DEFINITION. A torsion class is called stable provided it is closed under injective envelopes.

2.4 DEFINITION. We call a ring semi-perfect provided for each cyclic module $M \in {}_R\mathcal{M}$, there exists a projective module P_M, called the projective cover of M, such that $P_M \to M \to 0$ is exact and P_M is minimal in a technical sense (see Lambeck [6], p. 93, for an exact definition).

The following theorem will be used in proving succeeding theorems, but the proof is not meaningful in the present context. For a proof of this theorem see Rutter [7].

2.5 THEOREM. Let R be a semi-perfect ring and $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$ such that $\mathcal{F}$ is closed under homomorphic images. Then $\mathcal{T}$ is closed under projective covers.

2.6 THEOREM. Let R be a semi-perfect ring and let $(\mathcal{C},\mathcal{T},\mathcal{F})$ be a TTF theory for ${}_R\mathcal{M}$ such that $\mathcal{T}$ is stable. Then there exists a projective module P such that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$.

Proof: For each cyclic module $C \in \mathcal{C}$ there exists a projective cover P_C such that $P_C \to C \to 0$ is exact. Let $\mathcal{L}$ be a set of cyclic modules in $\mathcal{C}$ such that every cyclic module in $\mathcal{C}$ is isomorphic to one and only one module in $\mathcal{L}$, and let $P = \oplus\Sigma\{P_L \mid L \in \mathcal{L}\}$. Since $\mathcal{T}$ is closed under homomorphic images, $\mathcal{C}$ is closed under projective covers, so $P_L \in \mathcal{C}$ for each $L \in \mathcal{L}$. Since $\mathcal{C}$ is closed under arbitrary direct sums, $P \in \mathcal{C}$, and since each P_L is projective, P is projective. Let $\mathcal{T}' = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$. Clearly $\mathcal{T} \subseteq \mathcal{T}'$, since for each $T \in \mathcal{T}$, $\mathrm{Hom}(P,T) = 0$. Let $M \in \mathcal{T}'$, let $C \in \mathcal{C}$, let $f \in \mathrm{Hom}(C,M)$ and let $x \in f(C)$. Now $Rx \le f(C) \le M$. Since $\mathcal{T}$ is closed under injective envelopes, $\mathcal{C}$ is hereditary, so $Rx \le f(C)$ implies that $Rx \in \mathcal{C}$. Since Rx is cyclic there exists $L \in \mathcal{L}$ such that $L \simeq Rx$, so there exists $h \in \mathrm{Hom}(L,Rx)$ such that h is one-to-one and onto. Also, there exists a projective cover P_L of L and $g \in \mathrm{Hom}(P_L,L)$ such that g is onto. Let $\pi_L \in \mathrm{Hom}(P,P_L)$ be the natural projection. Then $hg\pi_L \in \mathrm{Hom}(P,Rx) \subseteq \mathrm{Hom}(P,M) = 0$, so $hg\pi_L = 0$. Then $0 = hg\pi_L(P) = hg(P_L) = h(L) = Rx$, so $x = 0$. Thus $f(C) = 0$, so $f = 0$, $\mathrm{Hom}(C,M) = 0$ for each $C \in \mathcal{C}$, and $M \in \mathcal{T}$. Therefore $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$. □

The next theorem provides a converse to Theorem 2.2 in a different context.

<u>2.7 THEOREM</u>. Let R be a semi-perfect ring, and let $(\mathcal{C},\mathcal{T},\mathcal{F})$ be a TTF theory for ${}_R\mathcal{M}$ such that $\mathcal{F}$ is closed under homomorphic images. Then there exists a projective module P such that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$.

<u>Proof</u>: For each cyclic module F in $\mathcal{F}$, there exists a projective cover P_F such that $P_F \to F \to 0$ is exact. Let $\mathcal{L}$ be a set of cyclic modules in $\mathcal{F}$ such that every cyclic module in $\mathcal{F}$ is isomorphic to one and only one element of $\mathcal{L}$, and let $P = \oplus\Sigma\{P_L \mid L \in \mathcal{L}\}$. Since each P_L is projective, P is projective. Now observe that $\mathcal{F} \subseteq \mathcal{C}$. To verify this, let $N \in \mathcal{F}$. Then there exists $N_c \leq N$ such that $N_c \in \mathcal{C}$ and $N/N_c \in \mathcal{T}$. Also, $N \to N/N_c \to 0$ is exact, so $N/N_c \in \mathcal{F}$, since $\mathcal{F}$ is closed under homomorphic images. Thus $N/N_c \in \mathcal{T} \cap \mathcal{F} = 0$, so $N/N_c = 0$, $N = N_c$, and $N \in \mathcal{C}$. Therefore $\mathcal{F} \subseteq \mathcal{C}$, and $L \in \mathcal{C}$ for each $L \in \mathcal{L}$. Since $\mathcal{T}$ is closed under homomorphic images, $\mathcal{C}$ is closed under projective covers, so $P_L \in \mathcal{C}$ for each $L \in \mathcal{L}$. Since $\mathcal{C}$ is closed under arbitrary direct sums, $P \in \mathcal{C}$. Let $\mathcal{T}' = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$. Clearly $\mathcal{T} \subseteq \mathcal{T}'$ since $\mathrm{Hom}(P,T) = 0$ for each $T \in \mathcal{T}$. Let $M \in \mathcal{T}'$, $N \in \mathcal{F}$ and $f \in \mathrm{Hom}(M,N)$. Let $x \in f(M)$. Then $Rx \leq f(M) \in \mathcal{F}$, so $Rx \in \mathcal{F}$. Since Rx is cyclic, there exists $L \in \mathcal{L}$ such that $L \simeq Rx$, so there exists $h \in \mathrm{Hom}(L,Rx)$ such that h is one-to-one and onto. Also,

there exists a projective cover P_L and $g \in \mathrm{Hom}(P_L, L)$ such that g is onto. Let $\pi_L \in \mathrm{Hom}(P, P_L)$ be the natural projection. Then $hg\pi_L \in \mathrm{Hom}(P, Rx)$. Let f' be the restriction of f to the submodule $f^{-1}(Rx)$. Since P is projective, there exists $\Phi \in \mathrm{Hom}(P, f^{-1}(Rx))$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[d, dashed, "\pi_L"] \arrow[dddl, dashed, "\Phi"'] & \\
 & P_L \arrow[d, dashed, "g"] & \\
 & L \arrow[d, dashed, "h"] & \\
f^{-1}(Rx) \arrow[r, dashed, "f'"'] & Rx \arrow[r, dashed] & 0
\end{tikzcd}$$

Now $\Phi \in \mathrm{Hom}(P, f^{-1}(Rx)) \subseteq \mathrm{Hom}(P,M) = 0$, so $\Phi = 0$. Then $0 = f'(0) = f'(\Phi(P)) = hg\pi_L(P) = hg(P_L) = h(L) = Rx$, so $Rx = 0$, $x = 0$, and $f(M) = 0$. Thus $f = 0$ and $\mathrm{Hom}(M,N) = 0$ for each $N \in \mathcal{F}$, so $M \in \mathcal{T}$ and $\mathcal{T}' \subseteq \mathcal{T}$. Therefore, $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(P,M) = 0\}$. □

A concluding theorem gives a characterization of a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ in terms of a module X uniquely determined by the elements of $F(\mathcal{T})$, and a corollary gives a similar characterization for a TTF class and its associated torsionfree class.

<u>2.8 THEOREM</u>. Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for ${}_R\mathcal{M}$. Let $X = \oplus\Sigma\{R/I \mid I \in F(\mathcal{T})\}$. Then $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \oplus\Sigma_A X \to M \to 0$ is exact for some index set A$\}$, and $\mathcal{F} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(X,M) = 0\}$.

<u>Proof</u>: Let $\mathcal{T}' = \{M \in {}_R\mathcal{M}\ \oplus\Sigma_A X \to M \to 0$ is exact for some index set A$\}$, and let $T \in \mathcal{T}$. Now for all $t \in T$, $(0:t) \in F(\mathcal{T})$, so $R/(0:t) \in$

$\{R/I \mid I \in F(T)\}$, and we can consider $R/(0:t)$ a submodule of X, as it is a direct summand. For each $I \in F(T)$, let n_I be the cardinal number of $\{t \in T \mid I = (0:t)\}$. Then let A be a set which has a cardinal number greater than or equal to n_I for each $I \in F(T)$. Let $\Phi \in \mathrm{Hom}(\oplus\Sigma_A X, \oplus\Sigma_T R/(0:t))$ be such that Φ is the identity mapping on $R/(0:t)$ for all $t \in T$ and is the zero mapping elsewhere. Thus, Φ is onto. Now $R/(0:t) \simeq Rt$ for each $t \in T$, so there exists an isomorphism $f\colon \oplus\Sigma_T R/(0:t) \to \oplus\Sigma_T Rt$. Also, there is a natural homomorphism $g\colon \oplus\Sigma_T Rt \to \Sigma_T Rt = T$ such that g is onto. Then $\oplus\Sigma_A X \xrightarrow{gf\Phi} T \to 0$ is exact, so $T \in T'$ and $T \subseteq T'$. Let $M \in T'$. Since $R/I \in T$ for all $I \in F(T)$ and T is closed under arbitrary direct sums, $X \in T$, and $\oplus\Sigma_A X \in T$ for any index set A. Since T is also closed under homomorphic images, $M \in T$. Thus $T' \subseteq T$ and $T = T'$.

Now we must show that $F = \{M \in {}_RM \mid \mathrm{Hom}(X,M) = 0\}$. Let $F' = \{M \in {}_RM \mid \mathrm{Hom}(X,M) = 0\}$. Since $X \in T$, $\mathrm{Hom}(X,F) = 0$ for all $F \in F$, so $F \subseteq F'$. Let $M \in F'$, $T \in T$, and $f \in \mathrm{Hom}(T,M)$. By the proof above, there exists an index set A such that $\oplus\Sigma_A X \xrightarrow{g} T \to 0$ is exact. Then $fg \in \mathrm{Hom}(\oplus\Sigma_A X, M) \simeq \Pi_A \mathrm{Hom}(X,M) = 0$, so $fg = 0$. Let $t \in T$. Since g is onto, there exists $x \in \oplus\Sigma_A X$ such that $g(x) = t$. Then $f(t) = f(g(x)) = fg(x) = 0$, so $f = 0$. Thus $\mathrm{Hom}(T,M) = 0$, $M \in F$ and $F' \subseteq F$. Therefore $F = F'$. □

<u>2.9 COROLLARY</u>. Let T be a TTF class, let I be the unique smallest element of $F(T)$, and let $X = R/I$. Then $T = \{M \in {}_RM \mid \oplus\Sigma_M X \to M \to 0$ is exact$\}$, and $F = \{M \in {}_RM \mid \mathrm{Hom}(X,M) = 0\}$.

<u>Proof</u>: Let $T' = \{M \in {}_RM \mid \oplus\Sigma_M X \to M \to 0$ is exact$\}$. Clearly $T' \subseteq T$, since $X \in T$. Let $T \in T$. For all $t \in T$, $(0:t) \in F(T)$, so $I \subseteq (0:t)$. Then $R/I \to R/(0:t) \to 0$ is exact for each $t \in T$, so there exists an onto

homomorphism Φ: $\oplus\Sigma_M X \to \oplus\Sigma_M R/(0{:}t)$. Also, as in the proof of Theorem 2.8, there exists $f \in \mathrm{Hom}(\oplus\Sigma_M R/(0{:}t), T)$ such that f is onto. Then $\oplus\Sigma_M X \xrightarrow{f\Phi} T \to 0$ is exact, so $T \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. Thus $\mathcal{T} = \mathcal{T}'$. The proof that $\mathcal{F} = \{M \in {}_R M \mid \mathrm{Hom}(X,M) = 0\}$ follows the pattern of the proof in Theorem 2.8. □

CHAPTER 3

TTF CLASSES

In Chapter One we defined torsion-torsionfree (TTF) classes, as well as TTF theories, and it was stated that if T is a TTF class, then the torsion filter $F(T)$ of T has a unique smallest element I. Furthermore, we will show that $F(T) = \{J \leq R \mid I \leq J\}$. Given a TTF theory (C,T,F), the following theorem characterizes C, T and F in terms of that unique smallest element of $F(T)$. These characterizations will prove to be quite useful in working with TTF classes.

<u>3.1 THEOREM</u>. Let (C,T,F) be a TTF theory, and let I be the unique smallest element of $F(T)$. Then,

(a) $T = \{M \in {}_RM \mid IM = 0\}$;

(b) $C = \{N \in {}_RM \mid IN = N\}$;

(c) $F = \{L \in {}_RM \mid \text{for all } 0 \neq x \in L,\ Ix \neq 0\}$.

<u>Proof</u>: (a) In the statement of Theorem 1.8, it was given that there is a one-to-one correspondence between the set of torsion filters for R and the set of hereditary torsion classes. Specifically, if B is a torsion filter for R, then the corresponding hereditary torsion class is $\{M \in {}_RM \mid (0:x) \in B \text{ for all } x \in M\}$. Then $T = \{M \in {}_RM \mid (0:x) \in F(T)$ for all $x \in M\} = \{M \in {}_RM \mid I \subseteq (0:x)$ for all $x \in M\} = \{M \in {}_RM \mid Ix = 0$ for all $x \in M\} = \{M \in {}_RM \mid IM = 0\}$.

(b) Let $C' = \{N \in {}_RM \mid IN = N\}$ and let $N \in C'$. Let $T \in T$ and let $f \in \mathrm{Hom}(N,T)$. Then $\mathrm{Im}f \leq T$, so $\mathrm{Im}f \in T$ since T is hereditary. By (a) above, $I \cdot \mathrm{Im} f = 0$. Then $0 = I \cdot f(N) = f(IN) = f(N)$, so $\mathrm{Im} f = 0$. Thus $f = 0$ and $N \in \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,T) = 0$ for all $T \in \mathcal{T}\} = \mathcal{C}$. Thus $\mathcal{C}' \subseteq \mathcal{C}$. Let $M \in \mathcal{C}$. Now $M \to M/IM \to 0$ is exact, so $M/IM \in \mathcal{C}$. Also, $I(M/IM) = 0$, so $M/IM \in \mathcal{T}$. Then $M/IM \in \mathcal{C} \cap \mathcal{T} = 0$, so $M/IM = 0$ and $M = IM$. Thus $\mathcal{C} \subseteq \mathcal{C}'$, and $\mathcal{C} = \mathcal{C}'$.

(c) Let $L \in \mathcal{F}$. Then $0 = L_t = \{x \in L \mid (0:x) \in F(\mathcal{T})\} = \{x \in L \mid I \subseteq (0:x)\} = \{x \in L \mid Ix = 0\}$. Thus for all $0 \neq x \in L$, $Ix \neq 0$, and $\mathcal{F} = \{L \in {}_R\mathcal{M} \mid$ for all $0 \neq x \in L$, $Ix \neq 0\}$. □

In Chapter Two, Theorem 2.2 gave a method for generating a TTF class with a projective module. The next theorem gives a method for generating a TTF class $\mathcal{T}$ when given an idempotent, two-sided ideal of R, and shows that this ideal will in fact be the unique smallest element of $F(\mathcal{T})$. It is then possible to demonstrate a one-to-one correspondence between the TTF classes for ${}_R\mathcal{M}$ and the idempotent, two-sided ideals of R.

<u>3.2 THEOREM</u>. Let I be an idempotent, two-sided ideal of R. Then $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid IM = 0\}$ is a TTF class and I is the unique smallest element of $F(\mathcal{T})$.

<u>Proof</u>: To verify that $\mathcal{T}$ is a TTF class, we must show that $\mathcal{T}$ is closed under (a) submodules, (b) homomorphic images, (c) extensions, (d) arbitrary direct products, and (e) arbitrary direct sums.

(a) Let $T \in \mathcal{T}$ and let $0 \to M \xrightarrow{f} T$ be exact. Now $I \cdot f(M) \subseteq I \cdot T = 0$, so $0 = I \cdot f(M) = f(IM)$. Since f is one-to-one, $IM = 0$. Then $M \in \mathcal{T}$ and $\mathcal{T}$ is closed under submodules.

(b) Let $T \in \mathcal{T}$ and let $T \xrightarrow{f} M \to 0$ be exact. Now $IM = I \cdot f(T) = f(IT) = f(0) = 0$, so $M \in \mathcal{T}$ and $\mathcal{T}$ is closed under homomorphic images.

(c) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be exact with A and C in $\mathcal{T}$. Let $b \in B$. Then $g(Ib) = Ig(b) \subseteq IC = 0$, so $Ib \subseteq \mathrm{Ker}\,g = \mathrm{Im}\,f$. Then $Ib \subseteq f(A)$ and $Ib = I(Ib) \subseteq If(A) = f(IA) = f(0) = 0$. Since the choice of b was arbitrary, $Ib = 0$ for all $b \in B$, so $IB = 0$ and $B \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under extensions.

(d) Let $\{T_i \mid i \in J\}$ be a collection of elements of $\mathcal{T}$. Then $I \cdot \Pi_J T_i = \Pi_J I \cdot T_i \simeq 0$, so $\Pi_J T_i \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary direct products.

(e) Let $\{T_i \mid i \in J\}$ be a collection of elements of $\mathcal{T}$. Then $\Pi_J T_i \in \mathcal{T}$ and $0 \to \oplus\Sigma_J T_i \to \Pi_J T_i$ is exact, so $\oplus\Sigma_J T_i \in \mathcal{T}$ by (a). Thus $\mathcal{T}$ is closed under arbitrary direct sums.

Therefore $\mathcal{T}$ is a TTF class. Now we must show that (f) $I \in F(\mathcal{T})$ and (g) I is the unique smallest element of $F(\mathcal{T})$.

(f) Since I is a two-sided ideal of R, $Ir \subseteq I$ for all $r \in R$. Then $I \cdot R/I = I\{r+I \mid r \in R\} = \{Ir+I \mid r \in R\} = I$. Since I is the zero element of R/I, R/I is thus in $\mathcal{T}$, and $I \in F(\mathcal{T})$.

(g) Let $J \in F(\mathcal{T})$. We first wish to show that $I \subseteq J$. Since $J \in F(\mathcal{T})$, $R/J \in \mathcal{T}$, so $I(R/J) = 0$. Then $0 = I(R/J) = IR+J/J = I+J/J$, so $J = I+J$ and $I \subseteq J$. If K is another smallest element of $F(\mathcal{T})$, $I \subseteq K$ and $K \subseteq I$, so $K = I$. Thus I is the unique smallest element of $F(\mathcal{T})$. □

It is to be noted here that $F(\mathcal{T}) = \{J \mid J \text{ is an ideal of } R \text{ and } I \subseteq J\}$. To verify this, observe that since I is the smallest element of $F(\mathcal{T})$, $I \subseteq J$ for all $J \in F(\mathcal{T})$, so $F(\mathcal{T}) \subseteq \{J \leq R \mid I \subseteq J\}$. Now if $J \leq R$ such that $I \subseteq J$, then $R/I \to R/J \to 0$ is exact. Since $R/I \in \mathcal{T}$ and $\mathcal{T}$ is closed under homomorphic images, $R/J \in \mathcal{T}$ and $J \in F(\mathcal{T})$. Thus $\{J \leq R \mid I \subseteq J\} \subseteq F(\mathcal{T})$ and $F(\mathcal{T}) = \{J \leq R \mid I \subseteq J\}$.

3.3 THEOREM. There is a one-to-one correspondence between the TTF classes for ${}_R\mathcal{M}$ and the idempotent, two-sided ideals of R.

Proof: Let π be a function mapping a TTF class $\mathcal{T}$ into the unique smallest element of its filter $F(\mathcal{T})$, and let δ be a function mapping an idempotent, two-sided ideal of R into its associated TTF class generated as in Theorem 3.2. We wish to show that $\delta\pi$ is the identity mapping on the set of TTF classes for ${}_R\mathcal{M}$ and $\pi\delta$ is the identity mapping on the set of idempotent, two-sided ideals of R. Let $\mathcal{T}$ be a TTF class and let I be the unique smallest element of $F(\mathcal{T})$. Then $\delta\pi(\mathcal{T}) = \delta(I) = \{M \in {}_R\mathcal{M} \mid IM = 0\} = \{M \in {}_R\mathcal{M} \mid I \subseteq (0:x) \text{ for all } x \in M\} = \{M \in {}_R\mathcal{M} \mid (0:x) \in F(\mathcal{T}) \text{ for all } x \in M\} = \mathcal{T}$.

Let I be an idempotent, two-sided ideal of R. We wish to show that $\pi\delta(I) = I$. First observe that if $\mathcal{T}$ is a TTF class, the unique smallest element of $F(\mathcal{T})$ is $\cap\{J \mid J \in F(\mathcal{T})\}$. Let $\mathcal{T}' = \{M \in {}_R\mathcal{M} \mid IM = 0\} = \delta(I)$, and let $I' = \pi\delta(I) = \pi(\mathcal{T}') = \cap\{J \mid J \in F(\mathcal{T}')\}$. Since $I(R/I) = 0$, $R/I \in \mathcal{T}'$, so $I \in F(\mathcal{T}')$ and $I' \subseteq I$. Now for all $J \in F(\mathcal{T}')$, $R/J \in \mathcal{T}'$, so $0 = I(R/J) = IR+J/J = I+J/J$, so $J = I+J$ and $I \subseteq J$. Thus $I \subseteq I'$ and $I = I'$. Therefore $\pi\delta(I) = I$. □

3.4 THEOREM. Let $(\mathcal{C},\mathcal{T},\mathcal{F})$ be a TTF theory for ${}_R\mathcal{M}$. Then the following are equivalent:

(a) $M = M_t \oplus M_c$ for all $M \in {}_R\mathcal{M}$;

(b) $R = R_t \oplus R_c$ (ring direct sum);

(c) $\mathcal{C} = \mathcal{F}$;

(d) $(M_c)_t = 0$ and $(M/M_t)_c = M/M_t$ for all $M \in {}_R\mathcal{M}$;

(e) $\mathcal{T}$ is closed under injective envelopes and R_c is a direct summand of R;

(f) $\mathcal{F}$ is closed under homomorphic images and R_t is a direct summand of R;

(g) R_c is a ring direct summand of R;

(h) $\mathcal{T}$ is closed under injective envelopes and $\mathcal{F}$ is closed under homomorphic images;

(i) R_t is a direct summand of R and $\mathcal{C}$ is closed under essential extensions.

<u>Proof</u>: The method of proof is to show (a) → (b) → (c) → (d) → (a); (b) and (c) → (e), (f), (g) and (h); (a) and (c) → (i); (e) → (b); (f) → (b); (g) → (b); (h) → (c); and (i) → (b).

(a→b) Assume $M = M_c \oplus M_t$ for all $M \in {}_R\mathcal{M}$. Since R is itself an R-module, $R = R_c \oplus R_t$, and since R_t and R_c are both two-sided ideals of R, this is a ring direct sum.

(b→c) Assume $R = R_c \oplus R_t$, where this is a ring direct sum. Since R_t and R_c are two-sided ideals of R, there exist orthogonal, central idempotents e_1 and e_2 of R such that $1 = e_1 + e_2$, $R_c = Re_1$ and $R_t = Re_2$. Since R_c is the unique smallest element of the torsion filter $F(\mathcal{T})$, we have that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid R_cM = 0\} = \{M \in {}_R\mathcal{M} \mid (Re_1M = 0\} = \{M \in {}_R\mathcal{M} \mid e_1M = 0\}$. Let $\mathcal{L} = \{M \in {}_R\mathcal{M} \mid R_tM = 0\} = \{M \in {}_R\mathcal{M} \mid e_2M = 0\}$. Observe that for all $T \in \mathcal{T}$ and $t \in T$, $e_2t = t$ since $e_2t = (1 - e_1)t = t - e_1t = t - 0 = t$. Similarly, for all $L \in \mathcal{L}$ and $\ell \in L$, $e_1\ell = \ell$. We now wish to show that $\mathcal{L} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,M) = 0 \text{ for all } T \in \mathcal{T}\} = \mathcal{F}$, and $\mathcal{L} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,T) = 0 \text{ for all } T \in \mathcal{T}\} = \mathcal{C}$.

First, let $L \in \mathcal{L}$, let $T \in \mathcal{T}$ and let $f \in \mathrm{Hom}(T,L)$. Now $f(T) \leq L$ and $e_2L = 0$, so $0 = e_2f(T) = f(e_2T) = f(T)$. Then $f = 0$, $\mathrm{Hom}(T,L) = 0$,

$L \in F$, and $L \subseteq F$. Now let $F \in F$. Since $e_2F \leq F$, then $e_1(e_2F) = (e_1e_2)F = 0 \cdot F = 0$, so $e_2F \in T \cap F = 0$. Thus $F \in L$, $F \subseteq L$, and $L = F$.

Let $L \in L$, let $T \in T$, and let $f \in \mathrm{Hom}(L,T)$. Now $f(L) \leq T$, so $f(L) \in T$ and $e_1f(L) = 0$. Then $0 = e_1f(L) = (1 - e_2)f(L) = f(L) - e_2f(L) = f(L) - f(e_2L) = f(L) - f(0) = f(L)$, so $f = 0$, $\mathrm{Hom}(L,T) = 0$, $L \in C$, and $L \subseteq C$. Let $C \in C$, and define $f:C \to e_2C$ by $f(c) = e_2c$ for all $c \in C$. Now $e_1(e_2C) = (e_1e_2)C = 0 \cdot C = 0$, so $e_2C \in T$ and $\mathrm{Hom}(C,e_2C) = 0$, which implies that $f = 0$. Then $0 = f(C) = e_2C$, so $C \in L$, $C \subseteq L$, and $L = C$. Thus we have that $F = C$.

(c→d) Assume $F = C$ and let $M \in {}_RM$. Since $M_c \in C$, $M_c \in F$. Now $(M_c)_t \leq M_c$, and since F is closed under submodules, $(M_c)_t \in F$. Also, $(M_c)_t \in T$, so $(M_c)_t \in T \cap F = 0$ and $(M_c)_t = 0$. Now $M/M_t \in F = C$, so $M/M_t \in C$. Since $(M/M_t)_c$ is the largest C-torsion submodule of M/M_t, $M/M_t \subseteq (M/M_t)_c$. Also, $(M/M_t)_c \leq M/M_t$, so $(M/M_t)_c = M/M_t$.

(d→a) Assume $(M_c)_t = 0$ and $(M/M_t)_c = M/M_t$ for all $M \in {}_RM$. Let $M \in {}_RM$. Since $M_c \leq M_c + M_t$, the sequence $M/M_c \to M/M_c{+}M_t \to 0$ is exact. Then $M/M_c{+}M_t \in T$, since $M/M_c \in T$ and T is closed under homomorphic images. Similarly, $M_t \leq M_c + M_t$, so $M/M_t \to M/M_c{+}M_t \to 0$ is exact. Since $M/M_t = (M/M_t)_c \in C$ and C is closed under homomorphic images, $M/M_c{+}M_t \in C$. Then $M/M_c{+}M_t \in C \cap T = 0$, so $M/M_c{+}M_t = 0$ and $M = M_c + M_t$.

Now $M_c \cap M_t \in T$, since $M_c \cap M_t \leq M_t \in T$ and T is closed under submodules. Also, $M_c \cap M_t \leq M_c$, so $M_c \cap M_t$ is a T-torsion submodule of M_c. Since $(M_c)_t$ is the largest T-torsion submodule of M_c, $M_c \cap M_t \leq (M_c)_t = 0$, so $M_c \cap M_t = 0$ and $M = M_c \oplus M_t$.

Clearly (b) and (c) imply (e), (f), (g) and (h).

(a,c→i) Assume $M = M_c \oplus M_t$ for all $M \in {}_R\mathcal{M}$. Clearly R_t is then a direct summand of R. Let $C \in \mathcal{C}$ and let N be an essential extension of C. Assume $N \notin \mathcal{C}$. Then $N \neq N_c$, and since $N = N_t \oplus N_c$, $N_t \neq 0$, so N_t is a non-zero submodule of N. Since C is essential in N, $C \cap N_t \neq 0$. Now $C \cap N_t \leq N_t \in \mathcal{T}$ and $\mathcal{T}$ is closed under submodules, so $C \cap N_t \in \mathcal{T}$. Also, $C \cap N_t \leq C \in \mathcal{C} = \mathcal{F}$ and $\mathcal{F}$ is closed under submodules, so $C \cap N_t \in \mathcal{C}$. Then $C \cap N_t \in \mathcal{C} \cap \mathcal{T} = 0$. This contradicts $C \cap N_t \neq 0$, so it must be true that $N \in \mathcal{C}$ and $\mathcal{C}$ is closed under essential extensions.

(e→b) Assume $\mathcal{T}$ is closed under injective envelopes and R_c is a direct summand of R. Then $R = R_c \oplus I$ for some ideal I of R, and $\mathcal{C}$ is hereditary. Now $I \simeq R/R_c \in \mathcal{T}$, so $I \in \mathcal{T}$. Then $I \subseteq R_t$, since R_t is the largest $\mathcal{T}$-torsion submodule of R, so $R = R_c + R_t$. Since $R_c \cap R_t \leq R_c$ and $\mathcal{C}$ is hereditary, $R_c \cap R_t \in \mathcal{C}$. Also, $R_c \cap R_t \leq R_t$ and $\mathcal{T}$ is closed under submodules, so $R_c \cap R_t \in \mathcal{T}$. Then $R_c \cap R_t \in \mathcal{C} \cap \mathcal{T} = 0$, so $R_c \cap R_t = 0$ and $R = R_c \oplus R_t$. Since R_c and R_t are both two-sided ideals of R, this is a ring direct sum.

(f→b) Assume $\mathcal{F}$ is closed under homomorphic images and R_t is a direct summand of R. Then $R = R_t \oplus I$ for some ideal I of R. Now $R/I \simeq R_t \in \mathcal{T}$, so $R/I \in \mathcal{T}$. Then $I \in F(\mathcal{T})$ and $R_c \subseteq I$, since R_c is the unique smallest element of $F(\mathcal{T})$. Then $0 \to I/R_c \to R/R_c$ is exact and $R/R_c \in \mathcal{T}$, so $I/R_c \in \mathcal{T}$. Now $I \simeq R/R_t \in \mathcal{F}$, so $I \in \mathcal{F}$. Since $I \to I/R_c \to 0$ is exact and $\mathcal{F}$ is closed under homomorphic images, $I/R_c \in \mathcal{F}$. Then $I/R_c \in \mathcal{T} \cap \mathcal{F} = 0$ and $I = R_c$. Thus $R = R_c \oplus R_t$, and this is a ring direct sum.

(g→b) Assume R_c is a ring direct summand of R. Then $R = R_c \oplus I$ for some two-sided ideal I of R, and $R_c = Re$ for some central idempotent e in R. Now $I \simeq R/R_c \in \mathcal{T}$, so $I \in \mathcal{T}$. Then $I \subseteq R_t$, since R_t is the largest $\mathcal{T}$-torsion submodule of R, so $R = R_c + R_t$. Let $x \in R_t \cap R_c$. Then $x \in R_c$, so there exists $a \in R$ such that $x = ae$. Also, $x \in R_t$, so $(0:x) \in F(\mathcal{T})$. Since R_c is the unique smallest element of $F(\mathcal{T})$, $R_c \subseteq (0:x)$, and $0 = ex = xe = (ae)e = ae = x$, so $R_c \cap R_t = 0$. Thus $R = R_c \oplus R_t$ and this is a ring direct sum.

(h→c) Assume $\mathcal{T}$ is closed under injective envelopes and $\mathcal{F}$ is closed under homomorphic images. Since $\mathcal{T}$ is closed under injective envelopes, $\mathcal{C}$ is hereditary. Let $F \in \mathcal{F}$, $T \in \mathcal{T}$, and $f \in \mathrm{Hom}(F,T)$. Now $f(F) \leq T$, so $f(F) \in \mathcal{T}$ since $\mathcal{T}$ is closed under submodules. Also, $f(F)$ is a homomorphic image of F, so $f(F) \in \mathcal{F}$. Then $f(F) \in \mathcal{T} \cap \mathcal{F} = 0$, so $f(F) = 0$, $f = 0$, and $\mathrm{Hom}(F,T) = 0$ for every $T \in \mathcal{T}$. Thus $F \in \mathcal{C}$ and $\mathcal{F} \subseteq \mathcal{C}$. Let $C \in \mathcal{C}$, $T \in \mathcal{T}$, and $f \in \mathrm{Hom}(T,C)$. Since $f(T)$ is a homomorphic image of T and $\mathcal{T}$ is closed under homomorphic images, $f(T) \in \mathcal{T}$. Also, $f(T) \leq C$, so $f(T) \in \mathcal{C}$. Then $f(T) \in \mathcal{C} \cap \mathcal{T} = 0$, so $f(T) = 0$, $f = 0$ and $\mathrm{Hom}(T,C) = 0$ for every $T \in \mathcal{T}$. Thus $C \in \mathcal{F}$ and $\mathcal{C} \subseteq \mathcal{F}$. Therefore $\mathcal{F} = \mathcal{C}$.

(i→b) Assume R_t is a direct summand of R and $\mathcal{C}$ is closed under essential extensions. Then $R = R_t \oplus I$ for some ideal I of R. Now $R/I \simeq R_t \in \mathcal{T}$, so $R/I \in \mathcal{T}$. Then $I \in F(\mathcal{T})$ and $R_c \subseteq I$, since R_c is the unique smallest element of $F(\mathcal{T})$. Also, $I \simeq R/R_t \in \mathcal{F}$, so $I \in \mathcal{F}$. Let A be a non-zero submodule of I. Then there exists $x \in A$ such that $x \neq 0$. Now $R_c x \neq 0$ since $I \in \mathcal{F}$ and $\mathcal{F} = \{M \in {}_R\mathcal{M} \mid \text{for all } 0 \neq x \in M, R_c x \neq 0\}$. Since R_c is a two-sided ideal of R, $R_c x \subseteq R_c$, and $R_c x \subseteq A$ since A is a left R-module. Then $0 \neq R_c x \subseteq R_c \cap A$, so $R_c \cap A \neq 0$ and $R_c \trianglelefteq I$. Then I is an essential extension of R_c, so $I \in \mathcal{C}$. Since R_c is the largest $\mathcal{C}$-torsion submodule of R, $I \subseteq R_c$. Thus $I = R_c$ and $R = R_t \oplus R_c$, and this is a ring direct sum. □

3.5 DEFINITION. A torsion theory (T,F) is called centrally splitting provided T is a TTF class with an associated TTF theory (C,T,F) which satisfies any (and hence all) of the properties (a) - (i) above.

The next theorem gives a characterization of the torsion and torsionfree classes of a centrally splitting torsion theory, and it serves as a lemma when setting up a one-to-one correspondence between the set of central idempotents of R and the set of TTF theories for ${}_RM$.

3.6 THEOREM. Let (T,F) be an arbitrary torsion theory. Then (T,F) is a centrally splitting torsion theory if and only if there exists a unique central idempotent e of R such that $T = T_e$ and $F = F_e$, where $T_e = \{M \in {}_RM \mid (1-e)M = M\}$ and $F_e = \{N \in {}_RM \mid eN = N\}$.

Proof: $(\rightarrow)$ Assume (T,F) is centrally splitting. Then there exists an associated TTF theory (C,T,F) such that $R = R_c \oplus R_t$, and this is a ring direct sum. Then there exists a central idempotent e of R such that $R_c = Re$. Since R_c is the unique smallest element of the filter $F(T)$, $T = \{M \in {}_RM \mid R_cM = 0\} = \{M \in {}_RM \mid (Re)M = 0\} = \{M \in {}_RM \mid eM = 0\} = \{M \in {}_RM \mid (1-e)M = M\} = T_e$, and $F = C = \{N \in {}_RM \mid R_cN = N\} = \{N \in {}_RM \mid (Re)N = N\} = \{N \in {}_RM \mid eN = N\} = F_e$.

To verify that e is unique, assume also that $T = T_f$ for some central idempotent f of R. Since T is centrally splitting, $R = R_t \oplus R_c$. Also, $F_f = C$ so $fR_c = R_c$, and $F_e = C$, so $eR_c = R_c$. Then $eR = eR_t + eR_c = e(1-e)R_t + eR_c = 0 + eR_c = eR_c = R_c$, and $fR = fR_t + fR_c = f(1-f)R_t + fR_c = fR_c = R_c$, so $eR = fR$. Then there exist $x,y \in R$ such that $e = fx$

and $f = ey$. Now $ef = (fx)f = f(fx) = fx = e$, and $ef = e(ey) = ey = f$, so $e = f$, and e is unique.

($\rightarrow$) Assume that $(\mathcal{T},\mathcal{F})$ is a torsion theory and that there exists a unique central idempotent e of R such that $\mathcal{T} = \mathcal{T}_e$ and $\mathcal{F} = \mathcal{F}_e$. We wish to first show that $\mathcal{T}_e$ is a TTF class, and then show that $(\mathcal{T}_e,\mathcal{F}_e)$ is centrally splitting. To verify that $\mathcal{T}_e$ is a TTF class, we must whow that (a) $\mathcal{T}_e$ is hereditary, and (b) $\mathcal{T}_e$ is closed under arbitrary direct products.

(a) Let $T \in \mathcal{T}_e$ and let $0 \rightarrow M \xrightarrow{f} T$ be exact. Then $e \cdot f(M) \leq eT = e(1-e)T = 0 \cdot T = 0$, so $0 = e \cdot f(M) = f(eM)$. Since f is one-to-one, this implies that $eM = 0$, so $M = (1-e)M$ and $M \in \mathcal{T}_e$. Thus $\mathcal{T}_e$ is hereditary.

(b) Let $\{T_i | \ i \in I\}$ be a collection of elements in $\mathcal{T}_e$. Then $(1-e)\Pi_I T_i = \Pi_I(1-e)T_i = \Pi_I T_i$, so $\Pi_I T_i \in \mathcal{T}_e$ and $\mathcal{T}_e$ is closed under arbitrary direct products.

Therefore $\mathcal{T}_e$ is a TTF class. Then there exists a torsion class $\mathcal{C}_e$ such that $(\mathcal{C}_e,\mathcal{T}_e,\mathcal{F}_e)$ is a TTF theory. To verify that $(\mathcal{T}_e,\mathcal{F}_e)$ is centrally splitting, note that $R = Re \oplus R(1-e)$ and $R(1-e) = R_t$, so R_t is a direct summand of R. Clearly $\mathcal{F}_e$ is closed under homomorphic images, so centrally splitting follows from (f) of Theorem 3.4. $\square$

<u>3.7 THEOREM</u>. There exists a one-to-one correspondence between the set of central idempotents of R and the set of centrally splitting torsion theories for ${}_R\mathcal{M}$.

<u>Proof</u>: Let Φ be a function mapping a central idempotent e of R into its associated centrally splitting torsion theory $(\mathcal{T}_e,\mathcal{F}_e)$ defined as in Theorem 3.6, and let Ψ be a function mapping a centrally splitting torsion theory $(\mathcal{T},\mathcal{F})$ into the unique central idempotent e of R such that

$T = T_e$ and $F = F_e$. We wish to show that $\Phi\Psi$ is the identity mapping on the set of centrally splitting torsion theories for ${}_RM$, and that $\Psi\Phi$ is the identity mapping on the set of central idempotents of R.

Let e be a central idempotent of R. Then $\Psi\Phi(e) = \Psi((T_e, F_e)) = e'$, where e' is a unique central idempotent of R such that $T_e = T_{e'}$ and $F_e = F_{e'}$. Since e' is unique, we have that e = e', and $\Psi\Phi(e) = e$.

Let (T, F) be a centrally splitting torsion theory for ${}_RM$, and let e be the unique central idempotent of R such that $T = T_e$ and $F = F_e$. Then $\Phi\Psi((T,F)) = \Phi(e) = (T_e, F_e) = (T,F)$. □

SUMMARY

In conclusion, we began with S. E. Dickson's concept of a torsion theory for an arbitrary abelian category, and we specified this concept for the category ${}_R M$ of left R-modules over an arbitrary ring R. After providing a foundation of definitions and theorems, it was proved that a class of left R-modules is a hereditary torsion class if and only if it is generated by an injective module.

A method was given for generating a TTF class from a projective module. It was then proved that if R is a semi-perfect ring, a stable TTF class is generated by a projective module. It was also proved that if R is a semi-perfect ring and if the associated torsionfree class is closed under homomorphic images, then a TTF class is generated by a projective module. Some questions that remain unanswered are whether or not an arbitrary TTF class is generated by a projective module, or else, exactly what conditions would have to be placed upon a TTF class, or upon an arbitrary ring R, in order that the TTF class be generated by a projective module.

A characterization was given of a hereditary torsion class in terms of a module X uniquely determined by the elements of the associated torsion filter. Then it was shown that a TTF class can be generated from a two-sided, idempotent ideal of R, and that this ideal will be the unique smallest element of the associated torsion filter. A one-to-one correspondence between the two-sided, idempotent ideals of R and the TTF classes for ${}_R M$ was given.

The definition of a centrally splitting torsion theory was given in terms of a theorem listing equivalent properties of a special type of TTF class, and then a one-to-one correspondence was demonstrated between the central idempotents of R and the centrally splitting torsion theories for ${}_R M$. As a result, (C,T,F) is a centrally splitting TTF theory if and only if R_c is generated by a central idempotent of R. An open question is: What results are obtained if R_t is generated by a central idempotent of R?

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